with the experimental data. Figure 3 (a) presents the rate of mass transfer predicted from the ideal theory versus the measured rate of mass transfer, while Fig. 3 (b) gives a similar comparison of the experimental data and the results computed from equation (7) ($b = 0.6$). Figure 3 shows that while the empirical correlation proposed herein predicts a mass-transfer rate that is consistently less than that realized experimentally, it represents a substantial improvement over that obtainable from the ideal theory. Indeed, if radiation heat transfer had been accounted for it is likely the agreement between the computed results and the experimental data would have been considerably improved.

CONCLUSIONS

The empirical correlation proposed in [1] for liquid-film cooling mass transfer, accounting for film roughness and entrainment effects, is extended to include liquid films of arbitrary length. A favorable comparison between the predicted results and the experimental data of Kinney, Abramson and Sloop [2] and Emmons and Warner [3] demonstrates the utility of the mass transfer correlation for predictions over a considerably wider range of experimental parameters than those investigated in [1].

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THE CRITERION FOR VALIDITY OF THE FIN APPROXIMATION

MYRON LEVITSKY

Mechanical Engineering, City College of New York, New York, N.Y., U.S.A.

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NOMENCLATURE

A,	cross-section area of fin;
h,	longitudinal surface convective coefficient;
h_e,	end surface convective coefficient;
H,	dimensionless fin parameter mh_e/h;
J_0, J_1,	ordinary Bessel functions of zeroth and first order;
k,	thermal conductivity of fin material;
m,	dimensionless fin parameter $(hP/kA)^{\frac{1}{2}}$;
P,	perimeter of fin cross-section area;
r',	radial coordinate;
R',	outer radius of fin;
x', y',	transverse fin coordinates;
z',	longitudinal or axial fin coordinate;
T_f,	surrounding fluid temperature;
T_w,	base temperature of fin;
θ,	dimensionless temperature $(T - T_f)/(T_w - T_f)$.

INTRODUCTION

The analysis of temperature distribution and heat flux in fins customarily makes use of a one-dimensional fin approximation wherein temperature gradients in the direction normal to the convective surface are neglected. Many texts and references which discuss this topic [1–5], some recent, state that the fin approximation is valid when transverse dimensions are small compared to the length of the fin. Crank and Parker [6] have shown, however, that in thin plates with convective surfaces, temperature gradients in the direction of the thickness become negligible when the Biot number based upon the thickness is much less than unity. Irey [7] has carried out a numerical comparison of the fin approximation with the exact solution for a cylindrical fin of finite length, and concludes that the accuracy of the fin approximation depends primarily upon the magnitude of a Biot number based upon the outer radius of the fin.

This note demonstrates that the temperature distribution in a fin of constant cross-section predicted by the fin approximation results explicitly from the exact solution, when the Biot number based upon the smallest transverse dimension is much less than unity, and that the thickness to length ratio plays no role in this relation. To this end, the note discusses two representative examples, an infinitely long fin with a rectangular cross-section, and a cylindrical fin of finite length.

THE INFINITELY LONG RECTANGULAR FIN

The fin approximation predicts that the temperature distribution along the length of an infinitely long fin whose rectangular cross-section has dimensions $2X'$ by $2Y'$, if $Y' \ll X'$, is given by

$$(T - T_f)/(T_w - T_f) = e^{-mz'} \tag{1}$$

where z' is measured from the base of the fin, and in which the fin parameter m is found from $m^2 = hP/kA \approx h/kY'$. Primed coordinates denote physical lengths, while unprimed quantities indicate their non-dimensionalized equivalents.

The exact solution for the three dimensional temperature distribution in the fin is formulated in terms of the dimensionless temperature θ, and dimensionless space variables $x = hx'/k$, $y = hy'/k$, $z = hz'/k$. The heat conduction equation thus becomes

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = 0 \tag{2}$$

while the boundary conditions may be written

$$\begin{aligned} &\frac{\partial\theta}{\partial x} + \theta = 0, x = \pm X; \qquad \theta(x, y, 0) = 1 \\ &\frac{\partial\theta}{\partial y} + \theta = 0, y = \pm Y; \qquad \theta(x, y, \infty) = \text{finite}. \end{aligned} \tag{3}$$

The solution is found to be [8]

$$\theta(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \exp\left[-\sqrt{(\alpha_m^2 + \beta_n^2)}z\right] \cos\alpha_m x \cos\beta_n y \tag{4}$$

where the coefficients A_{mn} are given by the relation

$$A_{mn} = \frac{4\sin\alpha_m X \sin\beta_n Y}{(\alpha_m X + \sin\alpha_m X\cos\alpha_m X)(\beta_n Y + \sin\beta_n Y\cos\beta_n Y)} \tag{5}$$

and where the eigenvalues α_m and β_n must satisfy the equations

$$\cot\alpha_m X - \alpha_m = 0; \qquad \cot\beta_n Y - \beta_n = 0. \tag{6}$$

The cotangent terms of the preceding equations may be expanded in series, and the smallest roots approximated in each case by

$$\alpha_1 \approx 1/X; \qquad \beta_1 \approx 1/Y. \tag{7}$$

When Y is much smaller than one, for some number of the eigenvalues α_m

$$\beta_1^2 \approx 1/Y \gg 1/X \lesssim \alpha_m^2 \tag{8}$$

and for these values, the coefficients A_{m1} are approximated by

$$A_{m1} = \frac{\sin\alpha_m X}{\alpha_m X + \sin\alpha_m X\cos\alpha_m X}. \tag{9}$$

Under these conditions, the double summation of equation (5) is dominated by those terms for which $n = 1$, and the solution for $\theta(x, y, z)$ is approximately given by

$$\theta(x, z) = e^{-\beta_1 z}\sum_{m=1}^{\infty}\frac{2\sin\alpha_m X\cos\alpha_m x}{\alpha_m X + \sin\alpha_m X\cos\alpha_m X}. \tag{10}$$

The summation which remains is readily shown to be a Fourier series representation of $F(x) = 1$, $-X \leqslant x \leqslant X$, and is identically equal to unity. Since $\beta_1 z = mz'$, equation (10) becomes

$$(T - T_f)/(T_w - T_f) = e^{-mz'}. \tag{1}$$

This is identical to the solution given by the fin approximation.

Thus, in a fin with a rectangular cross-section, the presence of one transverse Biot number which is much less than unity insures that the exact solution for the temperature distribution can be reduced to the fin approximation. The fin length plays no role in this process.

THE CYLINDRICAL FIN OF FINITE LENGTH

For a cylindrical fin of outer radius R', length L', where the circumferential surface has a film coefficient h, the end surface h_e, the fin approximation predicts that the temperature distribution along the length of the fin is given by [2]

$$\frac{T - T_f}{T_w - T_f} = \frac{\cosh m(L' - z') + H\sinh m(L' - z')}{\cosh mL' + H\sinh mL'} \tag{11}$$

where $m^2 = 2h/kR'$, and $H^2 = h_e^2R'/2kh$. The exact solution for the temperature distribution $\theta(r, z)$ is determined by the conduction equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) + \frac{\partial^2\theta}{\partial z^2} = 0 \tag{12}$$

together with boundary conditions

$$\frac{\partial \theta}{\partial z} + \frac{h_e}{h}\theta = 0, z = L; \qquad \theta(r, 0) = 1$$
$$\frac{\partial \theta}{\partial r} + \theta = 0, r = R; \qquad \theta(0, z) = \text{finite} \qquad (13)$$

where dimensionless spatial variables are $r = hr'/k$ and $z = hz'/k$. The exact solution for θ may be found to be

$$\theta(r, z) = \sum_{n=1}^{\infty} \frac{2}{R\lambda_n(1 + \lambda_n^2)} \frac{J_0(\lambda_n r)}{J_1(\lambda_n R)} \cdot \left[\frac{\cosh \lambda_n(L - z) + (h_e/h\lambda_n)\sinh \lambda_n(L - z)}{\cosh \lambda_n L + (h_e/h\lambda_n)\sinh \lambda_n L}\right] \qquad (14)$$

with the eigenvalues λ_n determined by the equation

$$J_0(\lambda_n R) - \lambda_n J_1(\lambda_n R) = 0. \qquad (15)$$

Using a series representation for the Bessel functions [9]

$$J_v(\lambda_n r) = \sum_{j=0}^{\infty} \frac{(-1)^j (\lambda_n r)^{2j+v}}{2^{2j+v} j!(j+v)!} \qquad (16)$$

the first several terms of equation (15) in ascending powers of R show that

$$0 = 1 - \frac{\lambda_n^2 R}{2} - \frac{\lambda_n^2 R^2}{4} + \frac{\lambda_n^4 R^3}{16} + \frac{\lambda_n^4 R^4}{64} \dots \qquad (17)$$

When $R \ll 1$, the first and lowest eigenvalue is found to be given by $\lambda_1^2 = 2/R$, the second next larger one by $\lambda_2^2 = 4/R^2$, and so forth. Thus, the eigenvalues form a sequence of ascending magnitude in which the first eigenvalue is much smaller than any following. In equation (14), the factor in square brackets is a monotonically decreasing function of z which has its maximum value of unity at the base of the fin, so that it cannot radically affect any terms of the summation. In view of the relative magnitudes of the eigenvalues when R is small, the summation for θ is dominated by the first term of the summation. The temperature distribution may thus be approximated by

$$\theta(r, z) = \left\{\frac{2}{R\lambda_1(1 + \lambda_1^2)} \frac{J_0(\lambda_1 r)}{J_1(\lambda_1 R)}\right\} \cdot \frac{\cosh \lambda_1(L - z) + (h_e/h\lambda_1)\sinh \lambda_1(L - z)}{\cosh \lambda_1 L + (h_e/h\lambda_1)\sinh \lambda_1 L}. \qquad (18)$$

The factor in braces may be simplified further by retaining only the first term of each series representation of the Bessel functions. In addition, $\lambda_1 L = mL'$ and $h_e/h\lambda_1 = H$. Substitution of the preceding quantities into equation (18) yields

$$\frac{T - T_f}{T_w - T_f} = \frac{\cosh m(L' - z') + H \sinh m(L' - z')}{\cosh mL' + H \sinh mL'} \qquad (13)$$

which is the solution given by the fin approximation. Here, one sees explicitly that the ratio of the outer radius to the fin length R'/L plays no role in the requirement which reduces the exact solution to the fin approximation.

CONCLUDING REMARKS

The one-dimensional fin solution has been successfully employed despite some confusion concerning the circumstances under which it is appropriate. A brief numerical comparison indicates in part why this has been so. Table 1 shows, for the cylindrical fin, the approximation to the first eigenvalue, and tabulated values [8] of λ_1 and λ_2 for several values of the radial Biot number R. It is evident that the approximated values of λ_1 are quite accurate over a relatively large range of R, and that for the same range of R, the second eigenvalue is much larger than the first. These conditions are sufficient to insure the accuracy of the fin approximation.

Table 1

Biot number R	λ_1 approximate	λ_1 tabulated	λ_2 tabulated
0·01	14·14	14·12	383·4
0·10	4·47	4·42	38·6
0·20	3·16	3·09	19·4
0·50	2·00	1·88	7·9

With combinations of transverse dimensions, convective film coefficients, and thermal conductivities commonly entering into the design of fins, the Biot number is almost always sufficiently small to insure the accuracy of the fin approximation. The fact that the length has been much larger than the thickness has been irrelevant. However, if the same analysis is applied to problems in which the surface film coefficient becomes large, as in boiling, and the fin is made of a non-metallic material with low thermal conductivity, the fin approximation may become inaccurate even though the ratio of transverse thickness to length is small.

It is therefore concluded that the criterion for the validity of the fin approximation is not a small thickness to length ratio, but a Biot number based upon thickness which is much less than one.

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NOTE ON THE GENERAL SOLUTION OF THE TRANSFER PROCESSES IN FINITE CAPILLARY POROUS BODY

M. D. MIKHAILOV

Institute for Mechanical and Electrotechnical Engineering, Sofia, Bulgaria

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IN [1] it is shown that the temperature and moisture distributions in one-dimensional bodies (plate, cylinder and sphere) are linear combinations of the solutions of boundary-value problems of pure heat conduction if the boundary conditions are of the same type for both potentials. In the present study the same will be demonstrated for a finite homogeneous region of arbitrary geometry.

The process of drying is described by a Luikov's system of equations [2]

$$\frac{\partial\theta_1(M,Fo)}{\partial Fo} = \nabla^2\theta_1(M,Fo) - Ko^*\frac{\partial\theta_2(M,Fo)}{\partial Fo} \quad (1)$$

$$\frac{\partial\theta_2(M,Fo)}{\partial Fo} = Lu\nabla^2\theta_2(M,Fo) - LuPn\nabla^2\theta_1(M,Fo). \quad (2)$$

In above equations $\theta_1(M, Fo)$ denotes dimensionless temperature; $\theta_2(M, Fo)$, dimensionless mass-transfer potential; M, position of a point in finite region V; Fo, Fourier number; $Ko^* = \varepsilon Ko$; ε, phase change criterion; Ko, Kossovich number; Lu, Luikov number; Pn, Posnov number; ∇^2, the Laplacian [2].

The initial potentials are prescribed functions defined in

$$\theta_k(M,0) = f_k(M), \quad k = 1,2. \quad (3)$$

The boundary conditions are

$$A(N)\frac{\partial\theta_k(N,Fo)}{\partial n} + B(N)\theta_k(N,Fo) = \varphi_k(N,Fo), \qquad k = 1,2 \quad (4)$$

where n is outward normal of S; S, boundary of V; $A(N)$ and $B(N)$, prescribed boundary coefficient functions defined on S; N, position of a point in the surface S; $\varphi_k(N, Fo)$, source functions on S:

The solution of the Sturm–Liouville problem

$$\nabla^2\psi_i(M) + \mu_i^2\psi_i(M) = 0 \quad (5)$$

$$A(N)\frac{\partial\psi_i(N)}{\partial n} + B(N)\psi_i(N) = 0 \quad (6)$$

is granted for known.

For the solution of problem (1)–(4) it is convenient to apply a three dimensional finite integral transform [3, 4].

$$\bar{\theta}_{ki}(Fo) = \int_V \psi_i(M)\,\theta_k(M,Fo)\,\mathrm{d}V. \quad (7)$$

It follows from the orthogonality of the eigenfunctions